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1999 J. Phys. A: Math. Gen. 32 8507

(http://iopscience.iop.org/0305-4470/32/48/311)

J. Phys. A: Math. Gen. **32** (1999) 8507–8525. Printed in the UK PII: S0305-4470(99)07049-3

Covariant spinor representation of $iosp(d, 2/2)$ and quantization of the spinning relativistic particle

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Received 17 August 1999

Abstract. A covariant spinor representation of $iosp(d, 2/2)$ is constructed for the quantization of the spinning relativistic particle. It is found that, with appropriately defined wavefunctions, this representation can be identified with the state space arising from the canonical extended BFV-BRST quantization of the spinning particle with admissible gauge fixing conditions after a contraction procedure. For this model, the cohomological determination of physical states can thus be obtained purely from the representation theory of the $iosp(d, 2/2)$ algebra.

1. Introduction and main results

In recent work the question of the 'super-algebraization' of the Hamiltonian BRST-BFV [1–3] extended phase space quantization for gauge systems has been examined [4, 7]. Specifically, following earlier indications along these lines in the literature [8], it has been claimed that there are natural spacetime 'quantization superalgebras' which possess representations precisely mirroring the BRST-BFV construction in certain cases, namely relativistic particle systems and generalizations thereof, for which the relevant spacetime supersymmetries are the superconformal algebra $osp(d, 2/2)$ and its inhomogeneous extension [4–6], or the family $D(2, 1; \alpha)$ of exceptional superalgebras, in the $(1 + 1)$-dimensional case [7].

In a previous work on the scalar particle [4], supersymmetry was realized using the method of produced superalgebra representations. In the present paper this programme is continued with the examination of the covariant BRST-BFV quantization of the spinning particle model via a spin representation of $iosp(d, 2/2)$, and the sharpening of previous work via a covariant tensor notation for this extended spacetime supersymmetry. Our specific results, to be elaborated in the remainder of the paper, are as follows. In section 2, a space of covariant spinor superfields carrying an appropriate spin representation of $iosp(d, 2/2)$ is introduced, and its structure studied. The generators J_{MN} of $osp(d, 2/2)$ have orbital and spin components, associated respectively with standard configuration space coordinates and differentials X^M, $P_M = \partial_M$, and an extended (graded) Clifford algebra with generators Γ_N entailing both fermionic and bosonic oscillators. The mass-shell condition $P \cdot P - \mathcal{M}^2 = 0$ factorizes, allowing the Dirac condition $\Gamma \cdot \partial - \mathcal{M} = 0$ to be covariantly imposed at the $(d+2/2)$-dimensional level, effecting a decomposition of the representation space. At the same time, the Dirac wavefunctions split into upper and lower components, so that the $iosp(d, 2/2)$ algebra is effectively realized on $2^{d/2}$-dimensional Dirac spinors (over x^M, and subject to a certain differential constraint on P_-, deriving from the mass-shell condition).

0305-4470/99/488507+19$30.00

In section 3 a 'BRST operator' Ω is named as one of the nilpotent odd generators of the homogeneous superalgebra (a 'super-boost' acting between fermionic and light cone directions), relative to a choice of 'ghost number' operator within the $sp(2)$ sector. Correspondingly a 'gauge fixing fermion' $\mathcal{F}$ of opposite ghost number is identified (a 'supertranslation' generator), and physical Hamiltonian $H = -\{\mathcal{F}, \Omega\}$. Finally, the cohomology of Ω is constructed at arbitrary ghost number. It is found that the 'physical states' thus defined are precisely those wavefunctions which obey the conventional $((d-1)+1)$-dimensional Dirac equation, and moreover which have a fixed degree of homogeneity in the light cone coordinate p_+. Given that the P_- constraint already dictates the evolution of the Dirac spinors in the light cone time $x^- = \eta^{-+}x_+$, the analysis thus reveals that this 'superalgebraization' of the BRST-BFV quantization yields the correct spin-$\frac{1}{2}$ irreducible representation of the Poincaré algebra in $(d-1)+1$ dimensions, as carried on the space of covariant solutions of the massive Dirac equation.

As this construction has been obtained purely algebraically, without the use of a physical model, it is finally the task of section 4 to establish that the standard Hamiltonian BRST-BFV *ansatz*, applied to the spinning particle model [8–13], does indeed give rise to an identical state space structure. The only proviso on this statement turns out to be that the model's extended phase space should formally be modified by a contraction, or 'β-limit' [14] in order to identify the appropriate sector of the full phase space (for details see section 4.2).

In conclusion, the import of our programme, exemplified by the present case study, is an approach to covariant quantization of models with gauge symmetries via a cohomological realization of the appropriate space of irreducible representations of physical states (in the case of particle models on flat spacetime, the Poincaré algebra) through the construction of the correct BRST complex (in the present cases, as realized within the covariant representations of the 'quantization superalgebras' $osp(d, 2/2)$ and generalizations). Further concluding remarks, and prospects for future work, are given in section 5.

2. Covariant representations of $iosp(d, 2/2)$

2.1. Introduction and notation

The $iosp(d, 2/2)$ superalgebra is a generalization of $iso(d, 2)$. The supermetric η_{MN} we shall use throughout is made up of three parts; the first has block diagonal form with the entries being the Minkowski metric tensor of $so(d, 1)$ with -1 occurring d times,

$$\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, \ldots, -1).$$

The second part is off-diagonal and can be written

$$\eta_{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $a, b = \pm$, reflecting a choice of light cone coordinates in two additional bosonic dimensions, one spacelike and one timelike.

The final part corresponds to the Grassmann odd components and is the symplectic metric tensor

$$\eta_{\alpha\beta} = \varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Here Greek indices $\alpha, \beta, \ldots$ take values $1, 2$, whilst $\lambda, \mu, \nu, \ldots$ take values in the range $0, \ldots, d-1$, and Latin indices $a, b, c, \ldots$ range over $0, \ldots d-1, +, -$. The indices $M, N, \ldots$

cover all values, and thus run over $0,\ldots,d-1,+,-,1,2$. We also define graded commutator brackets as

$$[\![A_M, B_M]\!] = A_M B_N - [MN] B_N A_M$$
$$\{\!\{A_M, B_M\}\!\} = A_M B_N + [MN] B_N A_M$$

where the convention for the $[MN]$ sign factor is $[MN] = (-1)^{mn}$ (extended to $[MN][NP] = (-1)^{mn+np}$ as necessary). The grading factors are $m, n = 0$ for Minkowski and light cone indices $\mu, \nu, \ldots, \pm$ and $m, n = 1$ for symplectic indices $\alpha, \beta, \ldots$. With these index conventions the metric thus obeys $\eta_{MN} = [MN]\eta_{NM}$.

We define $J_{MN} = -[MN]J_{NM}$ as the generators of the $osp(d, 2/2)$ superalgebra, with commutation relations as follows [15]:

$$[\![J_{MN}, J_{PQ}]\!] = -\eta_{NQ} J_{MP} + [NP]\eta_{NP} J_{MQ} - [MN][MP]\eta_{MP} J_{NQ} + [PQ][MN][MQ]\eta_{MQ} J_{NP}. \tag{1}$$

The homogeneous even subalgebra is $so(d,2) \oplus sp(2,\mathbb{R})$ with $so(d,2)$ generated by $J_{\mu\nu} = -J_{\nu\mu}$, and $sp(2,\mathbb{R})$ by $J_{\alpha\beta} = J_{\beta\alpha}$. For clarity, we set $J_{\alpha\beta} \equiv K_{\alpha\beta} = K_{\beta\alpha}$. Likewise, the odd generators will be denoted $J_{\mu\alpha} \equiv L_{\mu\alpha}$ or $J_{\alpha\pm} \equiv L_{\alpha\pm}$. The inhomogeneous part $i(d, 2/2)$ consists of additional (super)translation generators P_M satisfying

$$[\![J_{MN}, P_L]\!] = \eta_{LN} P_M - [MN]\eta_{LM} P_N. \tag{2}$$

The $d+2$ even translations are $P_\mu, P_\pm$ acting in the $(d,2)$ pseudo-Euclidean space, and the two odd nilpotent supertranslations are $P_\alpha \equiv Q_\alpha$.

We consider a class of covariant spinor superfield representations of $iosp(d,2/2)$ (cf [4, 16]) acting on suitable spinor wavefunctions $\Psi(x^M)$ over $(d+2/2)$-dimensional superspace†, $(\mathcal{B}\otimes\mathcal{F}\otimes\mathcal{S})$.

The $osp(d,2/2)$ generators can be more explicitly written as

$$J_{MN} = J^{\mathcal{L}}_{MN} + J^{\mathcal{S}}_{MN} \tag{3}$$

where the orbital part is defined

$$J^{\mathcal{L}}_{MN} = X_M \partial_N - [MN] X_N \partial_M \tag{4}$$

with

$$\partial_N = P_N = \frac{\partial}{\partial X^N} = \left(\frac{\partial}{\partial X^\mu}, \frac{\partial}{\partial X^\pm}, \frac{\partial}{\partial X^\alpha}\right).$$

The spin part of (3) is

$$J^{\mathcal{S}}_{MN} = \tfrac{1}{4}[\![\Gamma_M, \Gamma_N]\!] \tag{5}$$

where Γ_M, Γ_N are generalized Dirac matrices. Of course both $J^{\mathcal{L}}$ and $J^{\mathcal{S}}$ fulfil the $osp(d,2/2)$ algebra.

The graded Clifford algebra with generators Γ_N, acting on the space‡ $(\mathcal{B}\otimes\mathcal{F})$, is defined through

$$\{\!\{\Gamma_M, \Gamma_N\}\!\} = \Gamma_M\Gamma_N + [MN]\Gamma_N\Gamma_M = 2[MN]\eta_{MN} \tag{6}$$

† $\mathcal{S}$ denotes the superfields over $(d+2/2)$-dimensional superspace $(x^M) = (x^\mu, x^\pm, \theta^\alpha)$, while $\mathcal{B}\otimes\mathcal{F}$ carries the graded Clifford algebra (see below).

‡ $\mathcal{B}$ is the bosonic part which carries the representation of ζ_α (see below), while the fermionic $\mathcal{F}$ carries the representation of the Dirac algebra $\gamma_\mu, \gamma_\pm$ and γ_5.

(if $M \neq N$ then we can write $\Gamma_M\Gamma_N = -[MN]\Gamma_N\Gamma_M$). Writing the Γ in compact form as $\Gamma_M = (\Gamma_\mu, \Gamma_+, \Gamma_-, \Gamma_\alpha)^T$, we have

$$\begin{aligned}\Gamma_\mu &= 1 \otimes \hat{\gamma}_\mu \otimes 1\\ \Gamma_\pm &= 1 \otimes \hat{\gamma}_\pm \otimes 1\\ \Gamma_\alpha &= \zeta_\alpha \otimes \hat{\gamma}_5 \otimes (-1)^z\end{aligned} \tag{7}$$

where

$$\begin{aligned}\hat{\gamma}_\mu &= \begin{pmatrix}\gamma_\mu & 0\\ 0 & -\gamma_\mu\end{pmatrix} \qquad & \hat{\gamma}_+ &= \sqrt{2}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\\ \hat{\gamma}_- &= \sqrt{2}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \qquad & \hat{\gamma}_5 &= \begin{pmatrix}\gamma_5 & 0\\ 0 & -\gamma_5\end{pmatrix}\end{aligned} \tag{8}$$

$(-1)^z$ is the parity operator for the θ and is such that $(-1)^z\theta_\alpha = -\theta_\alpha$; $(-1)^z\theta_\alpha\theta_\beta = \theta_\alpha\theta_\beta$, and is defined such that $z \equiv \theta^\alpha\partial_\alpha$; γ_5 is defined such that $\gamma_5^2 = \kappa_5(=\pm 1)$.

The definition of $J^{\mathcal{S}}_{\alpha\beta}$ leads to

$$\begin{aligned}4J^{\mathcal{S}}_{\alpha\beta} = [\![\Gamma_\alpha, \Gamma_\beta]\!] &= \zeta_\alpha\gamma_5\zeta_\beta\gamma_5(-1)^{2z}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \zeta_\beta\gamma_5\zeta_\alpha\gamma_5(-1)^{2z}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\\ &= \kappa_5\{\zeta_\alpha, \zeta_\beta\}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.\end{aligned} \tag{9}$$

Moreover, $\{\![\Gamma_\alpha, \Gamma_\beta]\!\} = -2\varepsilon_{\alpha\beta}$ and so we take

$$[\zeta_\alpha, \zeta_\beta] = -2\kappa_5\varepsilon_{\alpha\beta}. \tag{10}$$

In the appendix we provide a realization of ζ_α (with $\kappa_5 = 1$) in terms of a pair of bosonic oscillators with indefinite metric.

From (1), (2) it is easy to establish the invariance of the square of the momentum operator, namely

$$\begin{aligned}[\![J_{MN}, P^R P_R]\!] &= (\delta^R_N P_M - [MN]\delta^R_M P_N) + P^R[MR][RN](\eta_{RN}P_M - [MN]\eta_{RM}P_N)\\ &= 0.\end{aligned} \tag{11}$$

Thus the second-order Casimir is

$$P^M P_M \equiv P^\mu P_\mu + Q^\alpha Q_\alpha. \tag{12}$$

Similarly we get the required generalization of the Pauli–Lubanski operator $[J_{MN}, W^{ABC}W_{CBA}] = 0$, providing a fourth-order Casimir operator. For any vector operator V_A we have

$$[J_{MN}, V_A] = \eta_{AM}V_N - [MN]\eta_{AN}V_M \tag{13}$$

and similarly for any tensor operator V_{AB}, V_{ABC}: for example,

$$\begin{aligned}[J_{MN}, V_{ABC}] = \eta_{AM}V_{NBC} - [MN]\eta_{AN}V_{MBC} + [MA][AN](\eta_{BM}V_{ANC}\\ -[MN]\eta_{BN}V_{AMC}) + [MA][AN][MB][BN](\eta_{CM}V_{ABN} - [MN]\eta_{CN}V_{ABM}).\end{aligned} \tag{14}$$

From this we can calculate

$$[J_{MN}, V^{ABC}V_{CBA}] = \eta^{AD}\eta^{BE}\eta^{CF}[J_{MN}, V_{DEF}V_{CBA}] = 0. \tag{15}$$

If we define $V_{ABC} = W_{ABC} = P_A J_{BC} + [BC][CA]P_C J_{AB} + [BA][AC]P_B J_{CA}$ we get the required identity.

2.2. *Dirac condition and reduced realization of* $iosp(d, 2/2)$ *superalgebra*

In order to project out irreducible representations of the full superalgebra, we require the mass-shell condition (Klein–Gordon equation): in representation terms, a requirement for reducibility of the $iosp(d, 2/2)$ representation

$$(P^M \eta_{MN} P^N - \mathcal{M}^2)\Psi = 0. \tag{16}$$

However, using the Clifford algebra just defined we have $[\![J_{MN}, \Gamma^L P_L]\!] = 0$ and so we can covariantly impose the stronger Dirac condition,

$$\begin{aligned} 0 &= (P^M \eta_{MN} P^N - \mathcal{M}^2) \\ &= (P^M \Gamma_M + \mathcal{M})(P^M \Gamma_M - \mathcal{M}) \end{aligned}$$

i.e., taking for example the positive root,

$$(P^M \Gamma_M - \mathcal{M})\Psi = 0. \tag{17}$$

We now construct the explicit forms of the generators J_{MN} of $iosp(d, 2/2)$ within this decomposition of the full space. Expanding the sum in (17) gives

$$(\Gamma^\mu P_\mu + \Gamma^+ P_+ + \Gamma^- P_- + \Gamma^\alpha Q_\alpha - \mathcal{M})\Psi = 0$$

or, in the explicit form, writing Ψ as a two-component array $\Psi = \begin{pmatrix} \psi \\ \sqrt{2}\phi \end{pmatrix}$, gives

$$\begin{pmatrix} \gamma^\mu P_\mu + \zeta^\alpha (-1)^z \gamma_5 Q_\alpha - \mathcal{M} & \sqrt{2} P_+ \\ \sqrt{2} P_- & -[\gamma^\mu P_\mu + \zeta^\alpha (-1)^z \gamma_5 Q_\alpha + \mathcal{M}] \end{pmatrix} \begin{pmatrix} \psi \\ \sqrt{2}\phi \end{pmatrix} = 0.$$

From this, we get the rather useful expression

$$\phi = -\frac{1}{2P_+}(\gamma^\mu P_\mu + \zeta^\alpha (-1)^z \gamma_5 Q_\alpha - \mathcal{M})\psi \tag{18}$$

and so P_- can be written

$$P_- \psi = -\frac{1}{2P_+}(\gamma^\mu P_\mu + \zeta^\alpha (-1)^z \gamma_5 Q_\alpha + \mathcal{M})(\gamma^\nu P_\nu + \zeta^\beta (-1)^z \gamma_5 Q_\beta - \mathcal{M})\psi. \tag{19}$$

Simplifying this equation yields

$$\begin{aligned} P_- \psi &= -\frac{1}{2P_+}(P^2 + Q_\alpha \varepsilon^{\beta\alpha} Q_\beta \kappa_5 - \mathcal{M}^2)\psi \\ &= -\frac{1}{2P_+}(P^2 + Q^\alpha Q_\alpha \kappa_5 - \mathcal{M}^2)\psi \end{aligned} \tag{20}$$

which we shall later use as the Hamiltonian. This equation is basically the Klein–Gordon equation (see equation (16)) of the BFV quantized spinning relativistic particle model which will carry the representation.

We are now in a position to explicitly determine the generators of $osp(d, 2/2)$. We show below the process for calculating $J_{\alpha-}$, and then state without proof all other terms, with the understanding that the same process is repeated for each.

We have $J_{\alpha-} = J^{\mathcal{L}}_{\alpha-} + J^{\mathcal{S}}_{\alpha-}$, where $J^{\mathcal{L}}_{\alpha-} = X_\alpha P_- - X_- P_\alpha$ and

$$\begin{aligned} J^{\mathcal{S}}_{\alpha-} &= \tfrac{1}{4}[\![\Gamma_\alpha, \Gamma_-]\!] = \tfrac{1}{2}\Gamma_\alpha \Gamma_- \\ &= \frac{1}{2}\begin{pmatrix} \zeta_\alpha \gamma_5 (-1)^z & 0 \\ 0 & -\zeta_\alpha \gamma_5 (-1)^z \end{pmatrix}\begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{1}{\sqrt{2}}\zeta_\alpha \gamma_5 (-1)^z \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{21}$$

and so

$$J_{\alpha-}\Psi = \begin{pmatrix} X_\alpha P_- - X_- P_\alpha & \frac{1}{\sqrt{2}}\zeta_\alpha\gamma_5(-1)^z \\ 0 & -X_\alpha P_- + X_- P_\alpha \end{pmatrix}\begin{pmatrix} \psi \\ \sqrt{2}\phi \end{pmatrix}$$
$$= \begin{pmatrix} (X_\alpha P_- - X_- P_\alpha)\psi + \zeta_\alpha\gamma_5(-1)^z\phi \\ -(X_\alpha P_- - X_- P_\alpha)\sqrt{2}\phi \end{pmatrix}.$$

Substituting in from equation (18) gives

$$J_{\alpha-} = X_\alpha P_- - X_- P_\alpha - \frac{\zeta_\alpha}{2P_+}(\gamma_5(-1)^z\gamma\cdot P + \gamma_5(-1)^z\zeta^\beta(-1)^z\gamma_5 P_\beta - \gamma_5(-1)^z\mathcal{M})$$
$$= X_\alpha P_- - X_- P_\alpha - \zeta_\alpha\zeta_- \tag{22}$$

where

$$\zeta_- = \frac{1}{2P_+}(\gamma_5(-1)^z\gamma\cdot P + \zeta^\beta\kappa_5 P_\beta - \gamma_5(-1)^z\mathcal{M}). \tag{23}$$

The remaining generators are

$$\begin{array}{ll} J_{\mu-} = X_\mu P_- - X_- P_\mu - \zeta_\mu\zeta_- & J_{\mu\alpha} = X_\mu P_\alpha - X_\alpha P_\mu + \frac{\kappa_5}{2}\zeta_\mu\zeta_\alpha \\ J_{\mu\nu} = X_\mu P_\nu - X_\nu P_\mu - \frac{\kappa_5}{4}[\zeta_\mu,\zeta_\nu] & J_{+\mu} = X_+ P_\mu - X_\mu P_+ \\ J_{+\alpha} = X_+ P_\alpha - X_\alpha P_+ & J_{\alpha\beta} = X_\alpha P_\beta + X_\beta P_\alpha + \frac{\kappa_5}{4}\{\zeta_\alpha,\zeta_\beta\} \\ J_{+-} = X_- P_+ - X_+ P_- - \frac{1}{2} & \end{array} \tag{24}$$

where we have defined

$$\zeta_\mu = \frac{\gamma_\mu\gamma_5(-1)^z}{\kappa_5}. \tag{25}$$

From the above definition we can easily show that $[\zeta_\mu,\zeta_\nu] = -\frac{1}{\kappa_5}[\gamma_\mu,\gamma_\nu]$. The non-zero commutation relations between ζ_- and the remaining operators for $iosp(d,2/2)$ can be calculated and are

$$\{\zeta_\mu,\zeta_-\} = \left\{\left[\frac{\gamma_\mu\gamma_5(-1)^z}{\kappa_5}, \frac{1}{2P_+}(\gamma_5(-1)^z\gamma\cdot P + \kappa_5\zeta^\beta P_\beta - \gamma_5(-1)^z\mathcal{M})\right]\right\} \tag{26}$$
$$= \frac{1}{2\kappa_5 P_+}[\![\gamma_\mu\gamma_5, \gamma_5\gamma^\nu P_\nu]\!] - \frac{\mathcal{M}}{\kappa_5}[\![\gamma_\mu\gamma_5,\gamma_5]\!] \tag{27}$$
$$= \frac{P_\mu}{P_+} \tag{28}$$

and

$$\begin{array}{lll} \{\gamma_5,\zeta_-\} = -\frac{\kappa_5(-1)^z}{P_+}\mathcal{M} & \{\zeta_-,\zeta_-\} = \frac{P_-}{P_+} & [X_-,\zeta_-] = -\frac{\zeta_-}{P_+} \\ [X_\mu,\zeta_-] = -\frac{\kappa_5}{2P_+}\zeta_\mu & \{X_\alpha,\zeta_-\} = -\frac{\kappa_5\zeta_\beta}{2P_+} & [\zeta_-,\zeta_\alpha] = -\frac{Q_\alpha}{P_+}. \end{array} \tag{29}$$

In summary, the realization of $iosp(d,2/2)$ that we use is formulated in terms of the operators X^μ, $P_\mu = \frac{\partial}{\partial x^\mu}$, γ_μ, γ_5, together with $X^\alpha = \theta^\alpha$, $P_\alpha = Q_\alpha = \frac{\partial}{\partial\vartheta^\alpha}$, ζ_α, ζ_-, and $X_+ = \tau I$, $P_- = H$, P_+, X_-. The non-zero commutation relations amongst these variables are

$$\begin{array}{lll} [X_\mu, P_\nu] = -\eta_{\mu\nu} & \{\theta_\alpha, Q_\beta\} = \varepsilon_{\alpha\beta} & [X_-, P_+] = 1 \\ [X_-, P_-] = -P_+^{-1}P_- & [\theta_\alpha, P_-] = P_+^{-1}Q_\alpha & [X_\mu, P_-] = P_+^{-1}P_\mu \end{array} \tag{30}$$

and

$$\{\zeta_\mu,\zeta_\nu\} = -2\kappa_5\eta_{\mu\nu} \qquad [\zeta_\alpha,\zeta_\beta] = -2\kappa_5\varepsilon_{\alpha\beta}. \tag{31}$$

Note in the above that X_+ and P_- are no longer canonically conjugate when acting on the ψ part of the superfield.

We have now calculated the complete set of non-zero commutation relations between the operators X^M, P_N, ζ_M, γ_5 and have shown that they do indeed provide the correct realization of $iosp(d, 2/2)$ on the ψ superfields. Remarkably, *precisely* these operators will emerge as the raw material in the extended BFV-BRST Hamiltonian quantization of the relativistic spinning particle model (section 4 below). However, the algebraic setting already provides the means to complete the cohomological construction of physical states, as we now show.

3. Physical states

The physical states of a system can be determined by looking at the action of the BRST operator Ω and the ghost number operator $N_{\rm gh}$ upon arbitrary states ψ, ψ'. As is well known [14], the physical states obey the equations

$$\Omega\psi = 0 \qquad \psi \neq \Omega\psi' \qquad \text{and} \qquad N_{\rm gh}\psi = \ell\psi$$

for some eigenvalue ℓ, where Ω is the BRST operator, and $N_{\rm gh}$ is the ghost number. Therefore, in order to determine the physical states we shall fix Ω and $N_{\rm gh}$, and determine their actions upon an arbitrary spinor-valued superfield ψ.

Take two c-number $sp(2)$ spinors η^α, η'^α with the following relations:

$$\begin{aligned} &\eta^\alpha\eta_\alpha = 0 = \eta'^\alpha\eta'_\alpha \\ &\eta^\alpha\eta'_\alpha = 1 = -\eta'^\alpha\eta_\alpha. \end{aligned} \tag{32}$$

An example of two such spinors is

$$\eta = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \qquad \eta' = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}. \tag{33}$$

Below in the superfield expansions we use

$$\begin{aligned} &\theta_\eta = \eta^\alpha\theta_\alpha \qquad \theta'_\eta = \eta'^\alpha\theta_\alpha \\ &\chi_\eta = \eta^\beta\chi_\beta \qquad \chi'_\eta = \eta'^\beta\chi_\beta. \end{aligned} \tag{34}$$

The first of these pairs of definitions leads to

$$\frac{\partial}{\partial\theta^\alpha} = \frac{\partial\theta_\eta}{\partial\theta^\alpha}\frac{\partial}{\partial\theta_\eta} + \frac{\theta'_\eta}{\partial\theta^\alpha}\frac{\partial}{\partial\theta_{\eta'}} = -\eta_\alpha\frac{\partial}{\partial\theta_\eta} - \eta'_\alpha\frac{\partial}{\partial\theta'_\eta}$$

and therefore

$$\eta^\alpha\frac{\partial}{\partial\theta^\alpha} = -\frac{\partial}{\partial\theta'_\eta} \qquad \text{and} \qquad \eta'^\alpha\frac{\partial}{\partial\theta^\alpha} = -\eta'^\alpha\eta_\alpha\frac{\partial}{\partial\theta_\eta} = \frac{\partial}{\partial\theta_\eta}. \tag{35}$$

Choose the BRST operator† and gauge fixing operator as

$$\begin{aligned} &\Omega = \eta^\alpha L_{\alpha-} \\ &\mathcal{F} = \eta'^\alpha P_\alpha \end{aligned} \tag{36}$$

and consistently the ghost number operator $N_{\rm gh} \equiv \eta^\alpha\eta'^\beta K_{\alpha\beta}$ satisfies

$$[N_{\rm gh}, \Omega] = \Omega \qquad \text{and} \qquad [N_{\rm gh}, \mathcal{F}] = -\mathcal{F}$$

as required.

† The corresponding anti-BRST operator is $\bar{\Omega} = \eta'^\alpha L_{\alpha-}$.

3.1. Action of ghost number operator

Note that in our case

$$N_{\mathrm{gh}} = \eta^1\eta'^1 K_{11} + \eta^1\eta'^2 K_{12} + \eta^2\eta'^2 K_{22} + \eta^2\eta'^1 K_{21} = -\tfrac{1}{2}(K_{11} - K_{22}).$$

$K_{\alpha\beta}$ can be written $K_{\alpha\beta} = K^S_{\alpha\beta} + K^L_{\alpha\beta}$ where the two parts denote the bosonic and fermionic (spin and orbital) contributions, respectively. This leads to N_{gh} having two parts as well:

$$K^L_{\alpha\beta} = \theta_\alpha \frac{\partial}{\partial\theta^\beta} + \theta_\beta \frac{\partial}{\partial\theta^\alpha} \tag{37}$$

therefore

$$N^L_{\mathrm{gh}} = \theta_\eta \frac{\partial}{\partial\theta_\eta} - \theta'_\eta \frac{\partial}{\partial\theta'_\eta} \tag{38}$$

and for the bosonic sector

$$K^S_{\alpha\beta} = \tfrac{1}{4}[\![\Gamma_\alpha, \Gamma_\beta]\!] = \tfrac{1}{4}\{\zeta_\alpha, \zeta_\beta\}\kappa_5 \tag{39}$$

therefore

$$\begin{aligned} N^S_{\mathrm{gh}} &= \frac{\kappa_5}{4}((\eta\cdot\zeta)(\eta'\cdot\zeta) + (\eta'\cdot\zeta)(\eta\cdot\zeta)) \\ &= \frac{\kappa_5}{2}(\eta\cdot\zeta)(\eta'\cdot\zeta) + \frac{\kappa_5}{4} = \frac{\kappa_5}{2}(\eta'\cdot\zeta)(\eta\cdot\zeta) - \frac{\kappa_5}{4}. \end{aligned} \tag{40}$$

As seen in the appendix, we can write a series expansion of an arbitrary spinor superfield ψ over $(x^\mu, x^\pm, \theta^\alpha)$ in an occupation number basis in the indefinite metric space acted on by ζ_α,

$$\begin{aligned} \psi &= \sum_{m,n=0}^{\infty} \psi^{(m,n)}|m,n\rangle = \sum_{m,n=0}^{\infty} (A^{(m,n)} + \theta^\gamma \chi^{(m,n)}{}_\gamma + \tfrac{1}{2}\theta^2 B^{(m,n)})|m,n\rangle \\ &= A + \theta^\alpha\chi_\alpha + \tfrac{1}{2}\theta^2 B. \end{aligned} \tag{41}$$

We can rewrite this series expansion with respect to the spinors (32) as follows:

$$\begin{aligned} \theta^\alpha\chi_\alpha &= \theta^\alpha\delta_\alpha^\beta\chi_\beta, = \theta^\alpha(\eta^\beta\eta'_\alpha - \eta_\alpha\eta'^\beta)\chi_\beta \\ &= \theta_\eta\chi'_\eta - \theta'_\eta\chi_\eta \end{aligned} \tag{42}$$

and

$$\begin{aligned} \tfrac{1}{2}\theta^2 &= \tfrac{1}{2}\theta^\alpha\varepsilon_{\alpha\beta}\theta^\beta = \tfrac{1}{2}\theta^\alpha(\eta_\alpha\eta'_\beta - \eta_\beta\eta'_\alpha)\theta^\beta \\ &= \tfrac{1}{2}(\theta_\eta\theta'_\eta - \theta'_\eta\theta_\eta) = \theta_\eta\theta'_\eta. \end{aligned} \tag{43}$$

Thus, using equations (42) and (43),

$$\psi^{(m,n)} = A^{(m,n)} + \theta_\eta\chi'^{(m,n)}_\eta - \theta'_\eta\chi^{(m,n)}_\eta + \theta_\eta\theta'_\eta B^{(m,n)}. \tag{44}$$

In what follows, the occupation number labels in the bosonic space will be suppressed, whereas the structure of the explicit superfield expansion will be needed. Thus, for example, $A \equiv \sum_{m,n=0}^{\infty} A^{(m,n)}$. Note that

$$\begin{aligned} &N^L_{\mathrm{gh}}A = 0 \qquad N^L_{\mathrm{gh}}(\theta_\eta\chi'_\eta) = \theta_\eta\chi'_\eta \\ &N^L_{\mathrm{gh}}(\theta_\eta\theta'_\eta B) = 0 \qquad N^L_{\mathrm{gh}}(-\theta'_\eta\chi_\eta) = \theta'_\eta\chi_\eta \end{aligned} \tag{45}$$

and so

$$\begin{aligned} N_{\mathrm{gh}}\psi = {} & \frac{1}{2}\left(\kappa_5(\eta\cdot\zeta)(\eta'\cdot\zeta) + \frac{\kappa_5}{2}\right)A + \frac{1}{2}\theta_\eta\left(\kappa_5(\eta\cdot\zeta)(\eta'\cdot\zeta) + \frac{\kappa_5+4}{2}\right)\chi'_\eta \\ & - \frac{1}{2}\theta'_\eta\left(\kappa_5(\eta\cdot\zeta)(\eta'\cdot\zeta) + \frac{\kappa_5-4}{2}\right)\chi_\eta + \frac{1}{2}\theta_\eta\theta'_\eta\left(\kappa_5(\eta\cdot\zeta)(\eta'\cdot\zeta) + \frac{\kappa_5}{2}\right)B. \end{aligned} \tag{46}$$

We demand that $N_{\mathrm{gh}}\psi = \ell\psi$ for some eigenvalue ℓ, therefore we can write

$$\kappa_5(\tfrac{1}{2}(\eta\cdot\zeta)(\eta'\cdot\zeta) + \tfrac{1}{4})A = \ell A$$

i.e.

$$\kappa_5(\eta\cdot\zeta)(\eta'\cdot\zeta)A = \left(\frac{4\ell-\kappa_5}{2}\right)A. \tag{47}$$

Similarly

$$\kappa_5(\eta\cdot\zeta)(\eta'\cdot\zeta)\chi'_\eta = \left(\frac{4\ell-\kappa_5-4}{2}\right)\chi'_\eta \tag{48}$$

$$\kappa_5(\eta\cdot\zeta)(\eta'\cdot\zeta)\chi_\eta = \left(\frac{4\ell-\kappa_5+4}{2}\right)\chi_\eta \tag{49}$$

$$\kappa_5(\eta\cdot\zeta)(\eta'\cdot\zeta)B = \left(\frac{4\ell-\kappa_5}{2}\right)B. \tag{50}$$

In the appendix the diagonalization of $(\eta\cdot\zeta)(\eta'\cdot\zeta)$ is carried out explicitly in the occupation number basis $\{|m,n\rangle\}$. Below we assume that suitable eigenstates can be found, and explore the consequences for the cohomology of the BRST operator at generic ghost number ℓ.

3.2. Action of BRST operator

The BRST charge is defined above as $\Omega = \eta^\alpha L_{\alpha-}$ and from section 2.2 we can write

$$L_{\alpha-} = \theta_\alpha P_- - X_-\frac{\partial}{\partial\theta^\alpha} - \zeta_\alpha\zeta_-. \tag{51}$$

We have previously (23) defined ζ_- and can write it as $\zeta_- = \frac{1}{2P_+}(D_5(-1)^z + \kappa_5\zeta^\beta P_\beta)$, where $D_5 = \gamma_5(\gamma\cdot P - \mathcal{M})$ is the Dirac operator multiplied by γ_5. Consequently,

$$\begin{aligned}\eta^\alpha\zeta_\alpha\zeta_- &= \frac{\eta^\alpha\zeta_\alpha}{2P_+}(D_5(-1)^z + \kappa_5\zeta^\beta P_\beta)\\ &= \frac{\eta^\alpha\zeta_\alpha(-1)^z D_5}{2P_+} + \kappa_5\frac{(\eta\cdot\zeta)(\zeta^\beta\frac{\partial}{\partial\theta^\beta})}{2P_+}.\end{aligned} \tag{52}$$

The second part of equation (52) can be further expanded as follows:

$$\zeta^\beta\frac{\partial}{\partial\theta^\beta} = \zeta_\gamma\frac{\partial}{\partial\theta^\beta}\varepsilon^{\beta\gamma} = \zeta_\gamma\frac{\partial}{\partial\theta^\beta}(-\eta^\beta\eta'^\gamma + \eta^\gamma\eta'^\beta) \tag{53}$$

which uses the identity $\varepsilon^{\beta\alpha} = (-\eta^\beta\eta'^\alpha + \eta^\alpha\eta'^\beta)$. Therefore,

$$\frac{(\eta\cdot\zeta)(\zeta^\beta\frac{\partial}{\partial\theta^\beta})}{2P_+} = \frac{1}{2P_+}(\eta\cdot\zeta)(\eta'\cdot\zeta)\frac{\partial}{\partial\theta'_\eta} + \frac{1}{2P_+}(\eta\cdot\zeta)^2\frac{\partial}{\partial\theta_\eta} \tag{54}$$

and so we can write

$$\eta^\alpha\zeta_\alpha\zeta_- = \frac{\eta^\alpha\zeta_\alpha(-1)^z D_5}{2P_+} + \frac{\kappa_5}{2P_+}(\eta\cdot\zeta)(\eta'\cdot\zeta)\frac{\partial}{\partial\theta'_\eta} + \frac{\kappa_5}{2P_+}(\eta\cdot\zeta)^2\frac{\partial}{\partial\theta_\eta}. \tag{55}$$

In a similar fashion we can expand the fermionic part of $L_{\alpha-}$ as follows:

$$\eta^\alpha L^L_{\alpha-} = \eta^\alpha\theta_\alpha P_- - \eta^\alpha X_-\frac{\partial}{\partial\theta^\alpha} = \theta_\eta P_- + \frac{\partial}{\partial\theta'_\eta}X_-$$

and

$$P_- = \frac{-1}{2P_+}((P^2-\mathcal{M}^2) + Q^\alpha Q_\alpha)$$

but

$$\varepsilon^{\beta\alpha} Q_\alpha Q_\beta = Q^\alpha Q_\alpha = (-\eta^\alpha \eta'^\beta + \eta^\beta \eta'^\alpha) \frac{\partial}{\partial\theta^\alpha} \frac{\partial}{\partial\theta^\beta}$$
$$= -2 \frac{\partial}{\partial\theta_\eta} \frac{\partial}{\partial\theta'_\eta}$$

therefore

$$P_- = \frac{-1}{2P_+} \left((P^2 - \mathcal{M}^2) + 2 \frac{\partial}{\partial\theta_\eta} \frac{\partial}{\partial\theta'_\eta} \right). \tag{56}$$

The BRST operator can thus be written

$$\Omega = \eta^\alpha L_{\alpha-} = -\theta_\eta \frac{P^2 - \mathcal{M}^2}{2P_+} - \frac{\theta_\eta}{P_+} \frac{\partial}{\partial\theta_\eta} \frac{\partial}{\partial\theta'_\eta} + \frac{\partial}{\partial\theta'_\eta} X_-$$
$$- \frac{\eta^\alpha \zeta_\alpha (-1)^z D_5}{2P_+} - \kappa_5 \frac{(\eta \cdot \zeta)(\eta' \cdot \zeta)}{2P_+} \frac{\partial}{\partial\theta'_\eta} - \kappa_5 \frac{(\eta \cdot \zeta)^2}{2P_+} \frac{\partial}{\partial\theta_\eta}. \tag{57}$$

By writing ψ as a series expansion to second order (equation (44)), we can determine the effect of Ω on ψ. For simplicity we shall write the effect of each term of Ω on ψ separately.
1st term:

$$-\theta_\eta \frac{P^2 - \mathcal{M}^2}{2P_+} \psi = -\theta_\eta \frac{P^2 - \mathcal{M}^2}{2P_+} A + \theta_\eta \theta'_\eta \frac{P^2 - \mathcal{M}^2}{2P_+} \chi_\eta.$$

2nd term:

$$-\frac{\theta_\eta}{P_+} \frac{\partial}{\partial\theta_\eta} \frac{\partial}{\partial\theta'_\eta} \psi = \frac{2\theta_\eta B}{2P_+}.$$

3rd term:

$$\frac{\partial}{\partial\theta'_\eta} X_- \psi = -X_- \chi_\eta - \theta_\eta X_- B.$$

4th term:

$$-\frac{\eta^\alpha \zeta_\alpha (-1)^z D_5}{2P_+} \psi = -(\eta \cdot \zeta) \frac{D_5}{2P_+} A + \theta_\eta (\eta \cdot \zeta) \frac{D_5}{2P_+} \chi'_\eta$$
$$-\theta'_\eta (\eta \cdot \zeta) \frac{D_5}{2P_+} \chi_\eta - \theta_\eta \theta'_\eta (\eta \cdot \zeta) \frac{D_5}{2P_+} B.$$

5th term:

$$-\kappa_5 \frac{(\eta \cdot \zeta)(\eta' \cdot \zeta)}{2P_+} \frac{\partial}{\partial\theta'_\eta} \psi = \frac{\kappa_5}{2P_+} (\eta \cdot \zeta)(\eta' \cdot \zeta) \chi_\eta + \kappa_5 \frac{\theta_\eta}{2P_+} (\eta \cdot \zeta)(\eta' \cdot \zeta) B.$$

6th term:

$$-\kappa_5 \frac{(\eta \cdot \zeta)^2}{2P_+} \frac{\partial}{\partial\theta_\eta} \psi = -\frac{\kappa_5}{2P_+} (\eta \cdot \zeta)^2 \chi'_\eta - \kappa_5 \theta'_\eta - \frac{\theta'_\eta}{2P_+} (\eta \cdot \zeta)^2 B.$$

Grouping $\Omega\psi$ with respect to coefficients of θ_η, θ'_η and $\theta_\eta \theta'_\eta$ we can write

$$\Omega\psi = C + C_{\theta_\eta} \theta_\eta + C_{\theta'_\eta} \theta'_\eta + C_{\theta_\eta \theta'_\eta} \theta_\eta \theta'_\eta \tag{58}$$

where we have

$$C = -X_-\chi_\eta - \eta\cdot\zeta\,\frac{D_5}{2P_+}A + \frac{\kappa_5}{2P_+}(\eta\cdot\zeta)(\eta'\cdot\zeta)\chi_\eta - \frac{\kappa_5}{2P_+}(\eta\cdot\zeta)^2\chi'_\eta \tag{59}$$

$$C_{\theta_\eta} = -\frac{P^2-\mathcal{M}^2}{2P_+}A + \frac{B}{P_+} - X_-B + \eta\cdot\zeta\,\frac{D_5}{2P_+}\chi'_\eta + \frac{\kappa_5}{2P_+}(\eta\cdot\zeta)(\eta'\cdot\zeta)B \tag{60}$$

$$C_{\theta'_\eta} = -\eta\cdot\zeta\,\frac{D_5}{2P_+}\chi_\eta - \kappa_5(\eta\cdot\zeta)^2\frac{B}{2P_+} \tag{61}$$

$$C_{\theta_\eta\theta'_\eta} = \frac{P^2-\mathcal{M}^2}{2P_+}\chi_\eta - \eta\cdot\zeta\,\frac{D_5}{2P_+}B. \tag{62}$$

Notice the apparent similarity between equations (61) and (62); these can in fact be shown to be a linear transformation of each other. Firstly, note that

$$D_5^2 = \gamma_5(\gamma\cdot P - \mathcal{M})\gamma_5(\gamma\cdot P - \mathcal{M}) = -\kappa_5(P^2-\mathcal{M}^2). \tag{63}$$

Thus, we can write equation (62) as

$$\begin{aligned} C_{\theta_\eta\theta'_\eta} &= -\frac{D_5^2}{2\kappa_5 P_+}\chi_\eta - \frac{(\eta\cdot\zeta)D_5}{2P_+}B \\ &= -\frac{D_5}{\kappa_5(\eta\cdot\zeta)}\left(\frac{(\eta\cdot\zeta)D_5}{2P_+}\chi_\eta + \kappa_5\frac{(\eta\cdot\zeta)^2}{2P_+}B\right). \end{aligned} \tag{64}$$

Thus, it can be seen that equations (61) and (62) differ only by a factor of $-D_5/(\kappa_5(\eta\cdot\zeta))$. Note that if

$$\frac{(\eta\cdot\zeta)D_5}{2P_+}\chi_\eta + \kappa_5\frac{(\eta\cdot\zeta)^2}{2P_+}B = 0$$

then both equations (61) and (62) will be zero. It is interesting to note that a similar situation exists in equations (59) and (60). The common component of these two equations is

$$\frac{D_5}{2P_+}A + \kappa_5\frac{(\eta\cdot\zeta)}{2P_+}\chi'_\eta.$$

Taking these similarities between the two pairs of equations into account we can redefine the expansion of ψ as follows: rescale χ_η and A by

$$\chi_\eta \equiv \tilde{\chi}_\eta + (\eta\cdot\zeta)\frac{D_5}{P^2-\mathcal{M}^2}B \tag{65}$$

$$A \equiv \tilde{A} + (\eta\cdot\zeta)\frac{D_5}{P^2-\mathcal{M}^2}\chi'_\eta \tag{66}$$

and so ψ becomes

$$\psi = \left(\tilde{A} + (\eta\cdot\zeta)\frac{D_5}{P^2-\mathcal{M}^2}\chi'_\eta\right) + \theta_\eta\chi'_\eta - \theta'_\eta\left(\tilde{\chi}_\eta + (\eta\cdot\zeta)\frac{D_5}{P^2-\mathcal{M}^2}B\right) + \theta_\eta\theta'_\eta B. \tag{67}$$

Using this redefinition equation (61) becomes

$$\begin{aligned} C_{\theta'_\eta} &= -\eta\cdot\zeta\,\frac{D_5}{2P_+}\tilde{\chi}_\eta - \frac{(\eta\cdot\zeta)^2}{2P_+}\frac{D_5^2}{P^2-\mathcal{M}^2}B - \kappa_5\frac{(\eta\cdot\zeta)^2}{2P_+}B \\ &= -\eta\cdot\zeta\,\frac{D_5}{2P_+}\tilde{\chi}_\eta + \frac{(\eta\cdot\zeta)^2}{2P_+}\frac{\kappa_5(P^2-\mathcal{M}^2)}{P^2-\mathcal{M}^2}B - \kappa_5\frac{(\eta\cdot\zeta)^2}{2P_+}B \\ &= -\eta\cdot\zeta\,\frac{D_5}{2P_+}\tilde{\chi}_\eta. \end{aligned} \tag{68}$$

By enforcing $\Omega\psi = 0$ we get $C_{\theta'_\eta} = 0$, which by (68) gives the Dirac equation

$$-\eta \cdot \zeta \frac{D_5}{2P_+}\tilde{\chi}_\eta = 0. \tag{69}$$

Similarly we can rewrite equation (62) as

$$C_{\theta_\eta\theta'_\eta} = \frac{P^2 - \mathcal{M}^2}{2P_+}\tilde{\chi}_\eta = \frac{1}{\kappa_5}D_5(D_5\tilde{\chi}_\eta) \tag{70}$$

which, by enforcing $\Omega\psi = 0$, leads to the Klein–Gordon equation

$$D_5^2\tilde{\chi}_\eta = 0. \tag{71}$$

Under the rescaling of equation (66), equation (59) becomes

$$C = \left(-X_- + \frac{\kappa_5}{2P_+}(\eta\cdot\zeta)(\eta'\cdot\zeta)\right)\chi_\eta - (\eta\cdot\zeta)\frac{D_5}{2P_+}\tilde{A} + \kappa_5\frac{(\eta\cdot\zeta)^2}{2P_+}\chi'_\eta - \kappa_5\frac{(\eta\cdot\zeta)^2}{2P_+}\chi'_\eta \tag{72}$$

and by substituting equation (49) into (72) we get

$$C = \left(-X_- + \frac{4\ell - \kappa_5 + 4}{4P_+}\right)\chi_\eta - (\eta\cdot\zeta)\frac{D_5}{2P_+}\tilde{A} = 0. \tag{73}$$

Similarly, using equations (66) and (50), (60) can now be written as

$$C_{\theta_\eta} = \frac{D_5^2}{2\kappa_5 P_+}\tilde{A} - \left(X_- - \frac{4\ell - \kappa_5 + 4}{4P_+}\right)B = 0 \tag{74}$$

where once again we have enforced the condition for physical states.

Defining the symbol

$$\tilde{X}_- = X_- - \frac{4\ell - \kappa_5 + 4}{4P_+} \tag{75}$$

and multiplying equation (74) by $(\eta\cdot\zeta)\frac{D_5}{P^2-\mathcal{M}^2}$ and then subtracting it from equation (73) we get

$$-\tilde{X}_-\left(\chi_\eta - (\eta\cdot\zeta)\frac{D_5}{P^2 - \mathcal{M}^2}B\right) = 0. \tag{76}$$

Substituting equation (65) into (76) gives the third equation of motion:

$$-\tilde{X}_-\tilde{\chi}_\eta = 0. \tag{77}$$

Finally, we identify the physical states at generic ghost number ℓ, arising from the cohomology of Ω, as the spinors $\tilde{\chi}$, with the following properties. From (69), the $\tilde{\chi}$ obey the usual massive Dirac equation. From (20), (56), the P_- constraint dictates the dependence of superfield components on light cone time $\tau = x^-$, as $P_- = \partial/\partial x^-$. Finally, interpreting (76), (77) in the p_+-representation (the Fourier transform of the x^+-representation, i.e. $X_- \equiv X^+ = -\partial/\partial p_+$), the $\tilde{\chi}$ are homogeneous functions of p_+ of degree $-(+1 - \frac{1}{4}\kappa_5 + \ell)$. Thus the $\tilde{\chi}$ are essentially only functions over $((d-1)+1)$-dimensional Minkowski space. For example, in the case $\kappa_5 = 1$, $\ell = -\frac{3}{4}$ (which implies $\Lambda = 0$), and using (A.16), we find that the physical states have the following explicit form†:

$$|\tilde{\chi}_\eta\rangle = \tilde{\chi}_\eta(x^\mu)[|0,0\rangle - |1,1\rangle + |2,2\rangle + \cdots + (-1)^m|m,m\rangle + \cdots] \tag{78}$$

in terms of the number states of the bosonic ghost sector (see the appendix), where $\tilde{\chi}_\eta(x^\mu)$ are ordinary functions of x^μ.

† $2\ell = -\frac{3}{2}$ is the canonical conformal dimension for a spinor field (see [17]).

4. BRST-BFV quantization of the spinning particle and $iosp(d, 2/2)$ structure

As is well known [14, 18], the BFV canonical quantization of constrained Hamiltonian systems [1–3] uses an extended phase space description in which, to each first class constraint ϕ_a, a pair of conjugate 'ghost' variables (of Grassmann parity opposite to that of the constraint) is introduced. Here we follow this procedure for the spinning relativistic particle. Although our notation is adapted to the massive case, $\mathcal{M} > 0$, as would follow from the second-order action corresponding to extremization of the proper length of the particle world line, an analysis of the *fundamental* Hamiltonian description of the first-order action [14] leads to an equivalent picture (with an additional mass parameter $\mu \neq 0$ supplanting m in the appropriate equations, and permitting $m \to 0$ as a smooth limit).

In either case, for the scalar or spinning particle the primary first class constraint is the mass-shell condition $\phi_1 = (P^2 - \mathcal{M}^2)$, where $P^2 = P^\mu \eta_{\mu\nu} P^\nu$. Including the Lagrange multiplier λ as an additional dynamical variable leads to a secondary constraint, reflecting conservation of its conjugate momentum π_λ. The quantum formulation should be consistent with the equations of motion and gauge fixing at the classical level, as such two restrictions are necessary so as to arrive at the particle quantization corresponding with the superalgebraic prescription of section 2. Firstly, we choose below to work in the class [19–22] $\dot{\lambda} = 0$; moreover, we take gauge fixing only to be with respect to gauge transformations in one of the *connected* components of the group, i.e. either the identity class, or the orientation reversing class characterized by $\tau' = \tau_i + \tau_f - \tau$. Thus λ will be quantized on the half-line (say $\mathbb{R}^+$), and the system is not modular invariant until the two distinctly oriented sectors (particle and anti-particle) are combined [14]. Secondly, we take $\phi_1^{(2)} = \lambda\pi_\lambda$ as the other secondary first class constraint (rather than $\phi_1^{(2)} = \pi_\lambda$ used in the standard construction)†. Finally, the spinning particle system also entails a second, Grassman odd first class constraint $\phi_2 = p_\mu\zeta^\mu + \mathcal{M}\epsilon\gamma_5 (\epsilon = \pm 1)$, together with its associated first class constraint $\phi_2^{(2)} = \pi_2^{(2)}$, the conjugate momentum of the corresponding Lagrange multiplier λ_2 (which is also Grassmann odd).

4.1. BFV extended state space and wavefunctions

The BFV extended phase space [14] for the BRST quantization of the spinning relativistic particle is taken to comprise the following canonical variables:

$$x^\mu(\tau), p_\mu(\tau), \zeta^\mu, \zeta_5, \lambda(\tau), \pi_\lambda(\tau), \lambda_2(\tau), \pi_2(\tau), \eta^{a(i)}, \rho_{a(i)}, a \qquad i = 1, 2. \tag{79}$$

$x^\mu(\tau), p_\mu(\tau)$ are Grassmann even whilst ζ^μ, ζ_5 are Grassmann odd variables, λ is the Grassmann even Lagrange multiplier corresponding to the even first class constraint ϕ_1, π_λ is the momentum conjugate to λ (which forms the constraint $\phi_1^{(2)}$), λ_2 is the odd Lagrange multipler corresponding to the Grassmann odd first class constraint ϕ_2, and $\phi_2^{(2)} = \pi_2^{(2)}$ is its conjugate momentum. $\eta^{1(1)}, \rho_{1(1)}$ and $\eta^{1(2)}, \rho_{1(2)}$ are the Grassmann odd conjugate pairs of ghosts corresponding to the constraints ϕ_1 and $\phi_1^{(2)}$ respectively, while $\eta^{2(1)}, \rho_{2(1)}$ and $\eta^{2(2)}, \rho_{2(2)}$ are the Grassmann even conjugate pairs of ghosts corresponding to the constraints ϕ_2 and $\phi_2^{(2)}$, respectively. We proceed directly to the quantized version by introducing the Schrödinger representation. We introduce operators X^μ, P_ν corresponding to the coordinates x^μ, p_ν, acting on suitable sets of wavefunctions over x^μ, and on the half-line $\lambda > 0$. The Hermitian ghosts $\eta^{a(i)}, \rho_{b(j)}$ (a pair of bc systems) are represented as usual either on a four-dimensional indefinite inner product space $|\sigma\sigma'\rangle, \sigma, \sigma' = \pm$, or here, in order to match with

† We thank J Govaerts for clarifying possible difficulties with regularity and independence of constraints.

section 2, in terms of suitable Grassmann variables acting on superfields. The non-zero commutation relations amongst (79) read (repeated in full for clarity):

$$\begin{aligned}&[X_\mu, P_\nu] = -\eta_{\mu\nu} \qquad \{\zeta_\mu, \zeta_\nu\} = -\tfrac{2}{\kappa_5}\eta_{\mu\nu} \qquad \{\gamma_5, \gamma_5\} = 2\kappa_5\\ &[\lambda, \pi_\lambda] = 1 \qquad \{\lambda_2, \pi_2\} = -1\\ &\{\eta^{1(i)}, \rho_{1(j)}\} = -\delta^i_j \qquad [\eta^{2(i)}, \rho_{2(j)}] = \delta^i_j \qquad i, j = 1, 2\end{aligned} \tag{80}$$

from which the algebra of constraints follows:

$$\{\phi_2, \phi_2\} = -2\kappa_5\phi_1\{\phi_1, \phi_2\} = \{\phi_1, \phi_1\} = 0. \tag{81}$$

The hermiticity conditions imposed on the above operators read

$$\begin{aligned}&X_\mu^\dagger = X_\mu \qquad P_\mu^\dagger = P_\mu \qquad \zeta_\nu^\dagger = \zeta_\nu \qquad \mu, \nu = 0, \ldots d-1\\ &\zeta_0^\dagger = -\zeta_0 \qquad \gamma_5^\dagger = \gamma_5\\ &\lambda^\dagger = \lambda \qquad \pi_\lambda^\dagger = \pi_\lambda \qquad \lambda_2^\dagger = -\lambda_2 \qquad \pi_2^\dagger = \pi_2\\ &(\eta^{a(i)})^\dagger = \eta^{a(i)} \qquad (\rho_{a(i)})^\dagger = (-1)^{a+1}(\rho_{a(i)}) \qquad a = 1, 2.\end{aligned} \tag{82}$$

The ghost number operator N_{gh} is defined by

$$N_{\mathrm{gh}} = \tfrac{1}{2}\sum_{a,i=1}^{2}(\eta^{a(i)}\rho_{a(i)} - (-1)^{(a-1)}\rho_{a(i)}\eta^{a(i)}). \tag{83}$$

The canonical BRST operator† is given by

$$\Omega = \eta^{1(1)}\phi_1 + \eta^{1(2)}\phi_1^{(2)} + \eta^{2(1)}\phi_2 + \eta^{2(2)}\phi_2^{(2)} + \tfrac{1}{2}(\eta^{2(1)})^2\rho_{1(1)}. \tag{84}$$

The gauge fixing operator [19] $\mathcal{F}$ which will lead to the appropriate effective Hamiltonian is given by

$$\mathcal{F} = -\tfrac{1}{2}\lambda\rho_{1(1)} \tag{85}$$

and thus the Hamiltonian can be written as

$$H = -[\![\mathcal{F}, \Omega]\!] = -\tfrac{1}{2}\lambda(\eta^{1(2)}\rho_{1(1)} + \phi_1) \tag{86}$$

which is, of course, BRST invariant.

Consider the following canonical transformations on the classical dynamical variables of the extended phase space [8]:

$$\left.\begin{aligned}\eta'^{a(i)} &= \lambda\eta^{a(i)}\\ \rho'_{a(i)} &= \tfrac{1}{\lambda}\rho_{a(i)}\end{aligned}\right\} a(i) = 1(1), 1(2), 2(1) \tag{87}$$

$$\lambda\pi'_\lambda = \lambda\pi_\lambda + (-\eta^{1(1)}\rho_{1(1)} + \rho_{1(2)}\eta^{1(2)} - \eta^{2(1)}\rho_{2(1)}) \tag{88}$$

with the remainder invariant. At the same time we relabel the coordinates $p_+ = \lambda^{-1}$ and $x_- = \lambda\pi_\lambda\lambda$. At the quantum level the corresponding BRST operator ($\Omega' = \eta'^{1(1)}\phi_1 + \eta'^{1(2)}\phi_1'^{(2)} + \eta'^{2(1)}\phi_2 + \eta^{2(2)}\phi_2^{(2)} + \frac{1}{2}(\eta'^{2(1)})^2\rho'_{1(1)}$) can be written as

$$\begin{aligned}\Omega' = {}&\lambda\eta^{1(1)}\phi_1 + \eta^{1(2)} : \lambda\phi_1^{(2)} : + \lambda\eta^{2(1)}\phi_2 + \eta^{2(2)}\phi_2^{(2)}\\ &-\lambda\eta^{1(2)}\eta^{1(1)}\rho_{1(1)} - \lambda\eta^{1(2)}\eta^{2(1)}\rho_{2(1)} + \tfrac{1}{2}\lambda(\eta^{2(1)})^2\rho_{1(1)}\end{aligned} \tag{89}$$

where the symmetric ordering

$$: \lambda\phi_1^{(2)} := \tfrac{1}{2}(\lambda\phi_1^{(2)} + \phi_1^{(2)}\lambda) = \lambda\phi_1^{(2)} - \tfrac{1}{2}\lambda$$

has been introduced.

† The criteria for the construction and nilpotency of the corresponding anti-BRST operator $\bar{\Omega}$ have been given in [23].

It is also convenient to define [8] the operators θ_α, Q_α, ζ'_α and $\tilde{\zeta}'_\alpha(\alpha = 1, 2)$ by

$$\begin{aligned} Q_{1,2} &= \tfrac{1}{2\sqrt{2}}(2\eta^{1(2)} \pm \rho_{1(1)}) \qquad & \theta_{1,2} &= \tfrac{1}{\sqrt{2}}(\pm\rho_{1(2)} - 2\eta^{1(1)}) \\ \zeta'_{1,2} &= \tfrac{1}{\sqrt{2}}(\eta^{2(1)} \pm \rho_{2(1)}) \qquad & \tilde{\zeta}'_{1,2} &= \tfrac{1}{\sqrt{2}}(\pm\eta^{2(2)} - \rho_{2(2)}) \end{aligned} \tag{90}$$

which obey the relations

$$\{Q_\alpha, \theta_\beta\} = \varepsilon_{\alpha\beta} \qquad \text{and} \qquad [\zeta'_\alpha, \zeta'_\beta] = -\varepsilon_{\alpha\beta}. \tag{91}$$

In terms of these variables we attain the following simple forms for the BRST, gauge fixing and Hamiltonian operators:

$$\begin{aligned} \Omega' &= \frac{1}{\sqrt{2}}(:\lambda\phi_1^{(2)}:(Q_1 + Q_2) + (\theta_1 + \theta_2)H + (\zeta'_1 + \zeta'_2)\lambda(\phi_2 + Q^\alpha\zeta'_\alpha) + (\tilde{\zeta}'_1 - \tilde{\zeta}'_2)\phi_2^{(2)}) \\ \mathcal{F}' &= -\frac{1}{2}\rho_{1(1)} = -\frac{1}{\sqrt{2}}(Q_1 - Q_2) \\ H' &= -[\![\mathcal{F}', \Omega']\!] = -\frac{\lambda}{2}(P^\mu P_\mu + Q^\alpha Q_\alpha - \mathcal{M}^2) \equiv H. \end{aligned} \tag{92}$$

Note that the ζ'_α defined here and the ζ_α defined in section 2.2 differ by a factor $\sqrt{2}$, i.e. $\zeta'_\alpha = \frac{1}{\sqrt{2}}\zeta_\alpha$.

4.2. *β-limiting procedure for the BRST operator*

It is now necessary to reconcile the development of sections 2 and 3, in which the identical raw material for construction of the BRST operator, gauge fixing function and hence physical states, appears *purely algebraically* (cf equations (30), (31), (29) with (80)) except for the absence of the $\eta^{2(2)}$, $\rho_{2(2)}$ even ghosts and thus the $\tilde{\zeta}_1$, $\tilde{\zeta}_2$ oscillators. In [8], a somewhat heuristic argument was provided to justify the restriction to the vacuum of the latter oscillators. Here instead we shall use what is known as the β-limiting procedure [14] applied throughout on the $a = 2$ label of the BFV phase space variables (if we also apply it to the $a = 1$ label we recover the Fadeev–Popov reduced phase space quantization scheme). The exposition will closely follow that of [14].

Consider, instead of (86), the gauge fixing fermion

$$\mathcal{F} = -\frac{1}{2}\lambda\rho_{1(1)} + \frac{1}{\beta}(\lambda_2 - \lambda_2^0)\rho_{2(2)} + \lambda_2\rho_{2(1)} \tag{93}$$

where β is arbitrary, real and Grassmann even and λ_2^0 is some given function of time with the same properties as λ_2. The Hamiltonian is thus given by

$$\begin{aligned} H_{\text{eff}} = [\![\mathcal{F}, \Omega]\!] = -\frac{1}{2}\lambda(\phi_1 + \eta^{1(2)}\rho_{1(1)}) + \frac{1}{\beta}(\lambda_2 - \lambda_2^0)\phi_2^{(2)} + \frac{1}{\beta}\eta^{2(2)}\rho_{2(2)} \\ +\eta^{2(2)}\rho_{2(1)} + \lambda_2\eta^{2(1)}\rho_{1(1)} + \lambda_2\phi_2. \end{aligned} \tag{94}$$

The equations of motion for the BFV phase space variables can be easily obtained for the above H by implementing as usual $\dot{A} = [\![A, H_{\text{eff}}]\!]$. We now change to new variables $\tilde{\pi}_2$, $\tilde{\rho}_{2(2)}$ such that $\pi_2 = \beta\tilde{\pi}_2$ and $\rho_{2(2)} = \beta\tilde{\rho}_{2(2)}$ and subsitute these into H_{eff}, Ω and N_{gh}, the equations of motion and the action related to H_{eff}. Having done that we take the limit $\beta \to 0$; in particular, the BRST and ghost number operators then become

$$\Omega = \eta^{1(1)}\phi_1 + \eta^{1(2)}\phi_1^{(2)} + \eta^{2(1)}\phi_2 + \tfrac{1}{2}(\eta^{2(1)})^2\rho_{1(1)} \tag{95}$$

$$N_{\text{gh}} = \sum_{i=1}^{2}\eta^{1(i)}\rho_{1(i)} + \eta^{2(1)}\rho_{2(2)} - \tfrac{1}{2} \tag{96}$$

while the equations of motion for $\lambda_2, \pi_2, \eta^{2(2)}$ and $\rho_{2(2)}$ (which are the ones that are affected by the β-limiting procedure) now become

$$(\lambda_2 - \lambda_2^0) = 0 \qquad \tilde{\pi}_2 = -\phi_2 - \eta^{2(1)}\rho_{1(1)} \qquad \eta^{2(2)} = 0 \qquad \rho_{2(2)} = -\tilde{\rho}_{2(2)}. \tag{97}$$

Solving these equations and taking $\lambda_2^0 = 0$, we find that equations (95) and (96) remain as they are whilst $H_{\text{eff}} = H$. Thus we have succeeded in 'squeezing out' the 2(2) pair of even ghosts together with the odd Lagrange multiplier λ_2. Moreover the Hamiltonian in equation (86), obtained from the admissible gauge fixing fermion given in equation (85), is recovered.

Finally, and most importantly, the canonical transformation in equations (87), (88) is not affected by this procedure, as can easily be observed. Thus whether we apply the canonical transformations before the β-limiting procedure or after does not matter. Consequently, via equations (87) and (88), equation (95) becomes

$$\Omega' = \frac{1}{\sqrt{2}}(:\lambda\phi_1^{(2)}: (Q_1 + Q_2) + (\theta_1 + \theta_2)H + (\zeta_1' + \zeta_2')\lambda(\phi_2 + Q^\alpha\zeta_\alpha)). \tag{98}$$

The forms (86) and (98) can now be shown to be identical to the previously given algebraically defined expressions for these quantities (20), (36). The raw material (30), (31), (29) also appears in this construction, as can be easily observed by (80), and by identifying $P_+ = \lambda^{-1}$, $X_- =: \lambda\phi_1^{(2)} :$, $\zeta_- = \phi_2 + Q^\alpha\zeta_\alpha'$ and the BRST operator $\Omega' = \eta^\alpha L_{\alpha-}$. Moreover, the realization of $iosp(d, 2/2)$ can be done as in (22), (24). In particular, the evaluation of the BRST cohomology performed in section 3 above, gives precisely the correct identification of physical state wavefunctions for the spinning particle model of this section, provided that we represent ζ_α' by $(-1)^z\zeta_\alpha'$ to account for the correct action on the superfield and the correct commutation relations of $iosp(d, 2/2)$. Also, the constant κ_5 appearing in (20) can also be introduced in the third equation of (92) to account for $\gamma_5 = \pm 1$, which will eventually appear in the factorization of P_- leading to the Dirac equation.

5. Conclusions

This paper, via the positive results claimed here for the test case of the spinning particle, provides confirmation of our programme of establishing the roots of covariant quantization of relativistic particle systems, in the BRST complex associated with representations of classes of extended spacetime supersymmetries. Similar examples under study are the '$D(2, 1; \alpha)$' particle in 1 + 1 dimensions [7], the higher spin-s case and the relation to Bargmann–Wigner equations, as well as considerations of how the method can be extended to, say, superstring or superparticle cases, for which a covariant approach has so far been problematical [24]. The general approach [5,6] is a classification of 'quantization superalgebras' in diverse dimensions, whose representation theory will implement the covariant quantization, in the spirit of the above example, of the appropriate classical phase space models of systems with gauge symmetries.

Conformal (super)symmetry has long been of interest as a probable higher symmetry underlying particle interactions, no more so than in the light of recent interpretations of compactifications of higher-dimensional supergravities [25, 26]. The present application is of particular interest in that the traditional descent from $d + 2$ to d dimensions—via a projective conformal space [17]—is here implemented not on the cone (massless irreps), but for the massive (super)hyperboloid. This paper can also be seen as an elaboration of the method of 'conformalization' [27], and as a version of 'two-time' physics [28–30]. Beyond the Dirac equation and higher spin generalizations, it will also be possible to investigate the algebraic BRST-BFV complex associated with indecomposable representations [31, 32] (for example, where the vector-scalar (super)special conformal generators are represented as nilpotent matrices).

Finally, it is important to point out that the present study has not attempted to settle the key question of the appropriate inner products for the covariant wavefunctions. Such further structure is under study, and can be expected to be important for the realization of modular invariance in the models. Further applications, such as the identification of the correct supermultiplets to which the scalar and Dirac propagators belong, will provide the rudiments of a theory of quantized fields at the $osp(d, 2/2)$ level.

Acknowledgments

The authors would like to thank Jan Govaerts for continuous encouragement and constructive suggestions and feedback on aspects of this work. The authors also thank Alex Kalloniatis and Peter West for discussions. Part of this work was gestated while PDJ was on study leave at NIKHEF, Amsterdam, and the hospitality of Jan Willem van Holten and the NIKHEF theory group is acknowledged. PDJ further acknowledges financial support from the Alexander von Humboldt Foundation. IT acknowledges the Australian Research Council for the award of a Fellowship, and Tony Bracken and the Centre for Mathematical Physics, University of Queensland, for support. Finally SC acknowledges support from an Australian Postgraduate Award.

Appendix. (a, b) representation of physical states

Appendix A.1. Preliminary construction

We can define a Heisenberg-like algebra as follows:

$$\begin{aligned}
&[a, b^\dagger] = 1 = [b, a^\dagger]\\
&[a, b] = 0 = [b, a]\\
&[a, a^\dagger] = 0 = [b, b^\dagger]
\end{aligned} \tag{A.1}$$

where we can take $a|0, 0\rangle = 0 = b|0, 0\rangle$, and define

$$|m, n\rangle = (a^\dagger)^m (b^\dagger)^n |0\rangle \qquad m, n \geqslant 0. \tag{A.2}$$

Note that this implies

$$\begin{aligned}
&\langle 0, 1|1, 0\rangle = \langle 0, 0|b^\dagger a|0, 0\rangle = 1\\
&\langle 1, 0|0, 1\rangle = 1.
\end{aligned} \tag{A.3}$$

In fact, in general we have

$$\langle m', n'|m, n\rangle = m!n!\delta_{m'n}\delta_{n'm} \tag{A.4}$$

and so we redefine our basis by

$$\begin{aligned}
&|m, n\rangle' = a^{\dagger m} b^{\dagger n}|0, 0\rangle\\
&|m, n\rangle = \frac{1}{\sqrt{m!n!}}|m, n\rangle' = \frac{(a^\dagger)^m (b^\dagger)^n}{\sqrt{m!n!}}|0, 0\rangle.
\end{aligned} \tag{A.5}$$

Appendix A.2. Realization of $\zeta_\alpha, \hat{\zeta}_\alpha$

As explained in section 2, the operators $\zeta_\alpha, \tilde{\zeta}_\alpha$ are constructed using a two-dimensional Bosonic oscillator algebra (a, b). We choose to define $\zeta_\alpha, \tilde{\zeta}_\alpha$ as follows:

$$\begin{aligned}
&\zeta_\alpha = \frac{1}{\sqrt{2}}((b \pm a) - (b^\dagger \mp a^\dagger))\\
&\tilde{\zeta}_\alpha = \frac{1}{\sqrt{2}}((a \pm b) - (a^\dagger \mp b^\dagger)).
\end{aligned} \tag{A.6}$$

At the same time we can define the ghost state η^α of section 4, and its conjugate momentum ρ_α as

$$\begin{aligned} \eta^1 &= \tfrac{\mathrm{i}}{\sqrt{2}}(a - a^\dagger) \qquad & \eta^2 &= \tfrac{\mathrm{i}}{\sqrt{2}}(b - b^\dagger) \\ \rho_1 &= \tfrac{1}{\sqrt{2}}(b + b^\dagger) \qquad & \rho_2 &= \tfrac{1}{\sqrt{2}}(a + a^\dagger). \end{aligned} \tag{A.7}$$

Equation (A.6), together with the spinors used in section 3 allows us to write

$$\begin{aligned} (\eta \cdot \zeta) &= \mathrm{i}(b - b^\dagger) = \sqrt{2}\eta_2 \\ (\eta' \cdot \zeta) &= -(a + a^\dagger) = -\sqrt{2}\rho_2. \end{aligned} \tag{A.8}$$

In section 3 the eigenstates of $(\eta \cdot \zeta)(\eta' \cdot \zeta)$, with eigenvalues ℓ were required in the analysis of the physical states. From equation (A.6) we can write

$$(\eta \cdot \zeta)(\eta' \cdot \zeta) = \mathrm{i}(a^\dagger b^\dagger - ab + ab^\dagger - a^\dagger b). \tag{A.9}$$

We have

$$a^\dagger b^\dagger |m, n\rangle = \sqrt{(m+1)(n+1)}|m+1, n+1\rangle \tag{A.10}$$

and

$$\begin{aligned} ab|m, n\rangle &= \frac{b^{\dagger m} a b^{\dagger n}}{\sqrt{m!n!}}|m+1, n+1\rangle \\ &= \frac{[b, a^{\dagger m}][a, b^{\dagger n}]}{\sqrt{m!n!}}|m, n\rangle \\ &= \sqrt{mn}|m-1, n-1\rangle. \end{aligned} \tag{A.11}$$

Similarly,

$$a^\dagger b|m, n\rangle = \frac{a^\dagger [b, a^{\dagger m}] a^{\dagger n}}{\sqrt{m!n!}}|m, n\rangle = m|m, n\rangle \tag{A.12}$$

$$ab^\dagger |m, n\rangle = \frac{a^{\dagger m}[a, b^{\dagger (n+1)}]}{\sqrt{m!n!}}|m, n\rangle = (n+1)|m, n\rangle. \tag{A.13}$$

As $(\eta \cdot \zeta)(\eta' \cdot \zeta)$ commutes with $(a^\dagger b - b^\dagger a)$ (the false ghost number), we specialize to eigenstates $|\Lambda\rangle$ with $m = n$

$$|\Lambda\rangle = \sum_0^\infty \Lambda_m |m, m\rangle. \tag{A.14}$$

Therefore,

$$\begin{aligned} (\eta \cdot \zeta)(\eta' \cdot \zeta)|\Lambda\rangle &= \mathrm{i}(a^\dagger b^\dagger - ab + ab^\dagger - a^\dagger b)|\Lambda\rangle \\ &= \mathrm{i}\sum_{m=0}^\infty [(m+1)\Lambda_m |m+1, m+1\rangle - \Lambda_m |m, m\rangle - m\Lambda_m |m-1, m-1\rangle] \\ &= \mathrm{i}[\Lambda_0|1, 1\rangle - \Lambda_0|0, 0\rangle + 2\Lambda_1|2, 2\rangle - \Lambda_1|1, 1\rangle - \Lambda_1|0, 0\rangle \\ &\quad + 3\Lambda_2|3, 3\rangle - \Lambda_2|2, 2\rangle - 2\Lambda_2|1, 1\rangle + \cdots] \\ &= \mathrm{i}[-(\Lambda_0 + \Lambda_1)|0, 0\rangle + (\Lambda_0 - \Lambda_1 - 2\Lambda_2)|1, 1\rangle + (2\Lambda_1 - \Lambda_2 - 3\Lambda_3)|2, 2\rangle \\ &\quad + \cdots + (m\Lambda_{m-1} - \Lambda_m - (m+1)\Lambda_{m+1})|m, m\rangle + \cdots] \\ &= \Lambda(\Lambda_0|0, 0\rangle + \Lambda_1|1, 1\rangle + \cdots + \Lambda_m|m, m\rangle + \cdots \end{aligned}$$

and so

$$\begin{aligned} -\mathrm{i}(\Lambda_0 + \Lambda_1) &= \Lambda\Lambda_0 \\ \mathrm{i}(\Lambda_0 - \Lambda_1 - 2\Lambda_2) &= \Lambda\Lambda_1 \\ \mathrm{i}(2\Lambda_1 - \Lambda_2 - 3\Lambda_3) &= \Lambda\Lambda_2 \end{aligned}$$

or in general

$$i(m\Lambda_{m-1} - \Lambda_m - (m+1)\Lambda_{m+1}) = \Lambda\Lambda_m. \tag{A.15}$$

Re-expressing these in terms of Λ and Λ_0 only we get

$$\begin{aligned}
\Lambda_1 &= (i\Lambda - 1)\Lambda_0 \\
\Lambda_2 &= -\tfrac{1}{2}(\Lambda^2 + 2i\Lambda - 2)\Lambda_0 \\
\Lambda_3 &= \tfrac{1}{6}(-i\Lambda^3 + 3\Lambda^2 + 8i\Lambda - 6)\Lambda_0 \\
\Lambda_4 &= \tfrac{1}{24}(\Lambda^4 + 4i\Lambda^3 - 20\Lambda^2 - 32i\Lambda + 24)\Lambda_0 \\
\Lambda_5 &= \frac{i}{120}(\Lambda^5 + 5i\Lambda^4 - 40\Lambda^3 - 100i\Lambda^2 + 184\Lambda + 120i)\Lambda_0
\end{aligned} \tag{A.16}$$

and so on.

It is easy to write a short program to generate Λ_m to any order.

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